

A NOTE ON HYPERBOLIC FLOWS IN SUB-RIEMANNIAN STRUCTURES

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ABSTRACT. The *curvature* and the *reduced curvature* are basic differential invariants of the pair (Hamiltonian system, Lagrange distribution) on the symplectic manifold. It is shown in [4] that the negativity of the reduced curvature implies the hyperbolicity of any compact invariant set of the Hamiltonian flow restricted to a prescribed energy level. We consider the Hamiltonian flows of the curve of least action of natural mechanical systems in sub-Riemannian structures with symmetries. We give sufficient conditions for the reduced flows (after reduction of the first integrals induced from the symmetries) to be hyperbolic and show new examples of Anosov flows. This result is a generalization of [8] and a partial generalization of [13] on magnetic flows.

1. INTRODUCTION

A prime example of Anosov flow is the geodesic flow on a compact Riemannian manifold with negative sectional curvature ([6]). It describes inertial motion of a point particle confined to the manifold. In this context, magnetic flows, the flows generated by special forces, were discussed more than 30 years ago by Anosov and Sinai [7]. They were studied recently by Gouda [8], Grognet [12], M. and P. Paternain [9] and M. P. Wojtkowski [13]. In the last reference, the potential and the so-called Gaussian thermostats of external fields were also considered. .

In the present note, we focus on the hyperbolicity of the flows associated with a natural mechanical system in a sub-Riemannian structure with multidimensional symmetries. In this case, the sub-Riemannian structures are reduced to a Riemannian manifold with a (vector-valued) magnetic field. We give sufficient conditions for the reduced Hamiltonian flows (after the reduction of the first integrals) to be hyperbolic in terms of the Riemannian curvature tensor and the magnetic field. As a consequence, a class of Anosov flows are also given.

In the second section, we formulate the main results of the note. We firstly introduce the notion of a dynamical Lagrangian distribution and then discuss the reduction after the first integrals. The key point is that we can construct the (reduced) curvature maps (forms) for the (reduced) dynamical Lagrangian distribution based on the work [14] and [3]. The negativity of reduced curvature forms implies the hyperbolicity of the Hamiltonian flows ([4]). Applying this criteria we give sufficient conditions for the reduced Hamiltonian flows to be hyperbolic, based on an expression of the reduced curvature forms via the Riemannian curvature tensor and the magnetic field.

The last section is devoted to the proofs of the main results. We apply the similar technique as in [10] to give the proof of the expression of the reduced curvature forms and the sufficient conditions of hyperbolic flows then easily follows.

2. MAIN RESULTS

2.1. Dynamical Lagrangian distributions. Let M be an even dimensional symplectic manifold endowed with a symplectic form σ . A *Lagrange distribution* $\Delta \subset TM$ is a smooth vector sub-bundle of TM such that each fiber $\Delta_x = \Delta \cap T_x M$, $x \in M$ is a Lagrangian subspace of the symplectic space $T_x M$. Basic examples are cotangent bundles endowed with the standard symplectic structure and the “vertical” distribution:

$$(1) \quad M = T^*N, \quad \Pi_x = T_x(T_q^*N), \quad \forall x = (p, q) \in T^*M, p \in T_q^*M, q \in M.$$

Let h be a Hamiltonian function on M and denote by \vec{h} the corresponding Hamiltonian vector field: $i_{\vec{h}}\sigma = dh$. We will assume that \vec{h} is a complete vector field without loss of generality since we will study

the dynamics of the Hamiltonian systems on a compact set. The pair (\vec{h}, Δ) will be said to be a *dynamical Lagrangian distribution* of the symplectic manifold (M, σ) .

Dynamical Lagrangian distributions appear naturally in Differential Geometry, Calculus of Variations and Rational Mechanics. The model example can be described as follows:

Example 1 On a manifold M for a given smooth function $L : TM \rightarrow \mathbb{R}$, which is convex on each fiber, we consider the following standard problem of Calculus of Variation with fixed endpoints q_0 and q_1 and fixed time T :

$$(2) \quad A(q(\cdot)) = \int_0^T L(q(t), \dot{q}(t)) dt \mapsto \min$$

$$(3) \quad q(0) = q_0, \quad q(T) = q_1.$$

Suppose that the Legendre transform $h : T^*M \rightarrow \mathbb{R}$ of the function L ,

$$(4) \quad h(p, q) = \max_{X \in T_q M} (p(X) - L(q, X)), \quad q \in M, p \in T_q^* M$$

is well defined and smooth on T^*M . We will say that the dynamical Lagrangian distributions (\vec{h}, Π) is associated with the problem (2)-(3), where Π is as in (1). \square

To describe the dynamical property of a dynamical Lagrangian distribution (\vec{h}, Δ) , we define the Jacobi curve (at point $x \in M$) of the pair (\vec{h}, Δ) as follows:

$$(5) \quad J_x(t) := e_*^{-t\vec{h}} (\Delta_{e^{t\vec{h}}x}),$$

where $e^{t\vec{h}}$, $t \in \mathbb{R}$ denotes the Hamiltonian flow generated by the vector field \vec{h} .

It is clear that the Jacobi curves $J_x(t)$ are curves in the Lagrange Grassmannian of the symplectic space T_x^*M . They are not arbitrary curves of the Lagrangian Grassmannian but inherit special features of the pair (\vec{h}, \mathcal{D}) . To specify these features recall that the tangent space $T_\Lambda L(W)$ to the Lagrangian Grassmannian $L(W)$ of a linear symplectic space W (endowed with a symplectic form ω) at the point Λ can be naturally identified with the space $\text{Quad}(\Lambda)$ of all quadratic forms on linear space $\Lambda \subset W$. Namely, given $\mathfrak{V} \in T_\Lambda L(W)$ take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$ and $\dot{\Lambda} = \mathfrak{V}$. Given some vector $l \in \Lambda$, take a curve $\ell(\cdot)$ in W such that $\ell(t) \in \Lambda(t)$ for all t and $\ell(0) = l$. Define the quadratic form

$$(6) \quad Q_{\mathfrak{V}}(l) = \omega(l, \frac{d}{dt}\ell(0)).$$

Using the fact that the spaces $\Lambda(t)$ are Lagrangian, it is easy to see that $Q_{\mathfrak{V}}(l)$ does not depend on the choice of the curves $\ell(\cdot)$ and $\Lambda(\cdot)$ with the above properties, but depends only on \mathfrak{V} . So, we have the linear mapping from $T_\Lambda L(W)$ to the spaces $\text{Quad}(\Lambda)$, $\mathfrak{V} \mapsto Q_{\mathfrak{V}}$. A simple counting of dimensions shows that this mapping is a bijection and it defines the required identification. A curve $\Lambda(\cdot)$ in a Lagrange Grassmannian is called *regular*, if its velocity is a nondegenerated quadratic form at every τ . A curve $\Lambda(\cdot)$ is called *monotone* (monotonically nondecreasing or monotonically nonincreasing) if the velocity is sign definite (nonnegative or nonpositive) at any point. For later convenience, a dynamical Lagrangian distribution is said to be regular (monotone) if the associated Jacobi curves are regular (monotone).

The group of symplectomorphisms of the ambient space acts naturally on Lagrangian distribution and Hamiltonian vector fields, therefore it acts also on dynamical Lagrangian distributions. It turns out ([14]) that one can construct the canonical bundle of moving frames and the complete system of symplectic invariants for parametrized curves in Lagrange Grassmannians satisfying very general assumptions (including monotone curves as a particular case). The complete system of symplectic invariants (value at $t = 0$) for the Jacobi curve $J_x(\cdot), x \in M$ is called the curvature maps of (\vec{h}, Δ) and it is the basic differential invariants of the pair (\vec{h}, Δ) w.r.t. the action of symplectic group of M . In this section, we will restrict us to the curvature maps for monotone regular dynamical Lagrangian distribution since our goal is to obtain a sufficient condition for hyperbolicity of the reduced Hamiltonian flows after the reduction of first integrals, while the reduced dynamical Lagrangian distributions are monotone regular (see Lemma 1 below). Note also that the curvature maps for regular curves in Lagrangian Grassmannians are constructed in earlier work [3].

More precisely, let $\mathfrak{R}_x(t)$ be the curvature map for the Jacobi curve $J_x(t), x \in M$. Then the linear map $\mathfrak{R}_x^{(\vec{h}, \Delta)} := \mathfrak{R}_x(0) = \mathfrak{R}_x(t)|_{t=0} : \Delta_x \rightarrow \Delta_x$ is said to be the *curvature map* (at x) of the dynamical Lagrangian distribution (\vec{h}, Δ) . It gives a symmetric bilinear forms (at x)

$$r_x^{(\vec{h}, \Delta)}(v, w) := \sigma(R_x^{(\vec{h}, \Delta)} w, [\vec{h}, V]), \quad v, w \in \Delta_x$$

where V is a smooth section of the sub-bundle Δ with $V(x) = v$. The corresponding quadratic form will be called the *curvature form* of the dynamical Lagrangian distribution (\vec{h}, Δ) .

Example 2 (Natural mechanical system) In Example 1, let

$$M = R^n, \Pi_{(p, q)} = (R^n, 0), L(q, X) = \frac{1}{2}|X|^2 - W(q)$$

(in this case the function $A(q(t), \dot{q}(t))$ is the Action functional of the natural mechanical system with potential energy $W(q)$). Then the curvature forms can be written as follows:

$$(7) \quad r_{(p, q)}^{(\vec{h}, \Pi)}(\partial_{p_i}, \partial_{p_j}) = \frac{\partial^2 W}{\partial q_i \partial q_j}(q), \quad \forall 1 \leq i, j \leq n.$$

In other words, in this case the curvature forms are naturally identified with the Hessian of the potential W . \square

Example 3 (Riemannian manifold) Let (M, g) be a Riemannian manifold. Let $L(q, X) = \frac{1}{2}g(X, X)$. The inner product $g(\cdot, \cdot)$ defines the canonical isomorphism between $T_q M$ and $T_q^* M$. For any $q \in M$ and $p \in T_q^* M$ we will denote by p^h the image of p under this isomorphism, namely, the vector $p^h \in T_q M$, satisfying

$$(8) \quad p(\cdot) = g(p^h, \cdot)$$

Since the fibers $T_q^* M$ are linear spaces, one can identify $\Pi_\lambda (= T_\lambda T_q^* M)$ with $T_{\pi(\lambda)}^* M$, i.e. the operation p^h is defined also on each $p \in \Pi_\lambda$ with values in $T_{\pi(\lambda)} M$. For any given $\lambda = (p, q) \in T^* M, p \in M, p \in T_q^* M$, it turns out ([3]) that

$$(9) \quad (\mathfrak{R}_\lambda^{(\vec{h}, \Pi)} v)^h = R^\nabla(p^h, v^h)p^h, \quad v \in \Pi_\lambda,$$

where R^∇ is the Riemannian curvature tensor of the metric g . \square

Example 4 (Natural mechanical system on a Riemannian manifold) We add the potential in the action functional in the previous example, i.e. $L(q, X) = \frac{1}{2}g(X, X) - W(q)$. Then the curvature maps satisfies

$$(10) \quad (\mathfrak{R}_\lambda^{(\vec{h}, \Pi)} v)^h = R^\nabla(p^h, v^h)p^h + \nabla_{v^h}(\nabla W)(q), \quad v \in \Pi_\lambda,$$

where ∇W is the gradient of W w.r.t. the Riemannian metric g . \square

2.2. Reduced curvature forms and hyperbolicity. Assume that the dynamical Lagrange distribution (\vec{h}, Δ) have arbitrary s first integrals g_1, \dots, g_s in involution with the Hamiltonian h , i.e. s functions on M such that

$$\{h, g_i\} = 0, \quad \{g_i, g_j\} = 0, \quad \forall 1 \leq i, j \leq s,$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. This problem appear naturally in the framework of mechanical systems and variational problems with symmetries. Let $\mathcal{G} = (g_1, \dots, g_s)$ and let

$$(11) \quad \Delta_x^\mathcal{G} = (\cap_{i=1}^s \ker d_x g_i) \cap \Delta_x + \text{span}\{\vec{g}_1(x), \dots, \vec{g}_s(x)\}$$

Clearly, the distribution $\Delta^\mathcal{G} = \{\Delta_x^\mathcal{G}, x \in M\}$ is a Lagrangian distribution. Hence, we get a reduced dynamical Lagrangian distribution $(\vec{h}, \Delta^\mathcal{G})$ after the reduction by first integrals \mathcal{G} . Its curvature maps (forms) will be called *the reduced curvature maps (forms)* after the reduction by first integrals \mathcal{G} .

Example 5 Assume that we have one first integral g of h such that the Hamiltonian vector field \vec{g} preserves the distribution Δ , i.e. $(e^{t\vec{g}})_* \Delta = \Delta$. Fixing some value c of g , one can define (at least locally) the following quotient manifold: $M_{g, c} = g^{-1}(c)/\mathcal{C}$, where \mathcal{C} is the line foliation of the integral curves of the vector field \vec{g} . The manifold $M_{g, c}$ naturally inherits a symplectic form from the original symplectic structure (M, σ) . Furthermore, if we denote by $\Phi : g^{-1}(c) \rightarrow M_{g, c}$ the canonical projection

on the quotient set, the vector field $\Phi_*(\vec{h})$ is well defined Hamiltonian vector field on $M_{g,c}$ due to the fact that the vector fields \vec{h} and \vec{g} commute. For simplicity, we still denote $\Phi_*(\vec{h})$ by \vec{h} . Actually we have simply described the standard reduction of the Hamiltonian systems on the level set of the first integrals in Mechanics (see e.g. [1]). In this way, to any dynamical Lagrangian distribution (\vec{h}, Δ) on M one can associate the dynamical Lagrangian distribution $(\vec{h}, \Phi_*\Delta)$ on the symplectic manifold $M_{g,c}$ of smaller dimension. \square

It is well known that the geodesic flows on a compact Riemannian manifold with negative sectional curvature is Anosov ([6]). On the other hand, the reduced curvature maps (forms) of the dynamical Lagrangian distributions associated with the geodesic problem on a Riemannian manifold are naturally identified with the sectional curvature tensor (See Example 6 below). Hence, we could roughly formulate the result of Anosov as follows: negativity of the reduced curvature forms implies the hyperbolicity of the geodesic flows. To go further from this viewpoint, one can obtain a natural generalization ([4]) in the framework of dynamical Lagrangian distributions. It will serve as a criteria in the study of the hyperbolic flows in sub-Riemannian structures.

Definition 1. Let e^{tX} , $t \in \mathbb{R}$ be the flow generated by the vector field X on a manifold P . A compact invariant set $A \subset P$ of the flow e^{tX} is called a hyperbolic set if there exists a Riemannian structure in a neighborhood of A , a positive constant δ , and a splitting: $T_z P = E_z^+ \oplus E_z^- \oplus \mathbb{R}X(z)$, $z \in A$ such that $X(z) \neq 0$ and

- (1) $e_*^{tX} E_z^+ = E_{e^{tX}z}^+$, $e_*^{tX} E_z^- = E_{e^{tX}z}^-$,
- (2) $\|e_*^{tX} \zeta^+\| \geq e^{\delta t} \|\zeta^+\|$, $\forall t > 0, \forall \zeta^+ \in E_z^+$,
- (3) $\|e_*^{tX} \zeta^-\| \leq e^{-\delta t} \|\zeta^-\|$, $\forall t > 0, \forall \zeta^- \in E_z^-$.

If the entire manifold P is a hyperbolic set, then the flow e^{tX} is called a flow of Anosov type.

Theorem 1. Let $c = (c_0, c_1, \dots, c_s)$ be constants. Let S be a compact invariant set of the flow $e^{t\vec{h}}$ contained in a fixed level of $h^{-1}(c_0) \cap_{i=1}^s g_i^{-1}(c_i)$ and $\vec{h}(x), \vec{g}_i(x) \notin \mathcal{D}_x, \forall x \in S, i = 1, \dots, s$. If the reduced curvature form $r_x^{(g,h)}$ of the dynamical Lagrangian distribution (\vec{h}, \mathcal{D}) (after the reduction by first integrals (\mathcal{G}, h)) is negative at every point x of S , then S is a hyperbolic set of the flow $e^{t\vec{h}}|_{h^{-1}(c_0) \cap_{i=1}^s g_i^{-1}(c_i)}$.

2.3. Descriptions of main results. We now specialize to the study of a natural mechanical system on a sub-Riemannian manifold with symmetries.

Let M be a connected smooth manifold. A distribution \mathcal{D} on M is a sub-bundle of the tangent bundle TM . It is said to be *completely nonholonomic* if any local frame $\{X_i : 1 \leq i \leq n\}$ for \mathcal{D} , together with all its iterated Lie brackets $[X_i, X_j], [X_i, [X_j, X_k]], \dots$, spans the tangent bundle TM . A Lipschitzian curve $\gamma : [0, T] \rightarrow M$ is said to be admissible if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for a.e. $t \in [0, T]$. From the Rashevskii-Chow theorem ([5]) it follows that there is an admissible curve joining any two points of M . A *sub-Riemannian metric* is a smoothly varying positive definite inner product $\langle \cdot, \cdot \rangle$ on \mathcal{D} . In particular, when \mathcal{D} is equal to the tangent bundle, $\langle \cdot, \cdot \rangle$ gives a Riemannian metric.

A *sub-Riemannian structure*, denoted by the triple $(M, \mathcal{D}, \langle \cdot, \cdot \rangle)$, is a smooth n -dimensional connected manifold M equipped with a sub-Riemannian metric $\langle \cdot, \cdot \rangle$ on a completely nonholonomic distribution \mathcal{D} . In this case, we call the manifold M a *sub-Riemannian manifold*. In the present note we consider sub-Riemannian metrics $\langle \cdot, \cdot \rangle$ on distribution \mathcal{D} of corank s , having s transversal infinitesimal symmetries, i.e. s vector fields X_1, \dots, X_s on M such that

$$(12) \quad e_*^{tX_i} \mathcal{D} = \mathcal{D}, \quad (e^{tX_i})^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle, \quad 1 \leq i \leq s,$$

and $TM = \mathcal{D} \oplus \text{span}\{X_1, \dots, X_s\}$. Suppose further that the symmetries $\{X_i : 1 \leq i \leq s\}$ are commutative (see Remark 1 for noncommutative case), i.e.

$$(13) \quad [X_i, X_j] = 0, \quad \forall 1 \leq i, j \leq s.$$

We consider the natural mechanical system on a sub-Riemannian manifold (ASR):

$$(14) \quad A(\gamma(\cdot)) = \int_0^T \left(\frac{1}{2} \|\dot{\gamma}\|^2 - W(\gamma) \right) dt \mapsto \min$$

$$(15) \quad \gamma(\cdot) \text{ is admissible, } \gamma(0) = q_0, \quad \gamma(T) = q_1.$$

where $\|\cdot\|$ is the norm w.r.t. the metric $\langle \cdot, \cdot \rangle$. We assume further that the potential W in (14) is constant along the integral curves of any X_i , or, equivalently,

$$(16) \quad X_i(W) = 0, \quad i = 1, \dots, s.$$

It is more convenient to regard it as an optimal control problem and its extremals can be described by the Pontryagin Maximum Principle of Optimal Control Theory ([11]). There are two different types of extremals: abnormal and normal, according to vanishing or nonvanishing of Lagrange multiplier near the functional, respectively. The minimizers of the problem are the projections of either normal extremals or abnormal extremals.

In the present note we will focus on normal extremals only. To describe them let us introduce some notations. Let T^*M be the cotangent bundle of M and σ be the canonical symplectic form on T^*M , i.e. $\sigma = -d\varsigma$, where ς is the tautological (Liouville) 1-form on T^*M . Let

$$(17) \quad h(p, q) = \max_{u \in \mathcal{D}} (p \cdot u - \frac{1}{2} \|u\|^2 + W(q)) = \frac{1}{2} \|p|_{\mathcal{D}_q}\|^2 + W(q), \quad q \in M, \quad p \in T_q^*M,$$

where $p|_{\mathcal{D}_q}$ is the restriction of the linear functional p to \mathcal{D}_q and the norm $\|p|_{\mathcal{D}_q}\|$ is defined w.r.t. the Euclidean structure on \mathcal{D}_q . It is well defined and smooth in the open set $O = T^*M \setminus \mathcal{D}^\perp$, where \mathcal{D}^\perp is the annihilator of \mathcal{D} , that is,

$$(18) \quad \mathcal{D}^\perp = \{(p, q) \in T^*M : p(v) = 0 \quad \forall v \in \mathcal{D}_q\}.$$

For any vector field X_i define the “quasiimpluses” $u_i : T^*M \rightarrow \mathbb{R}$ by

$$u_i(p, q) = p(X_i(q)), \quad q \in T_q^*M, q \in M, \quad \forall 1 \leq i \leq s.$$

Let h be the sub-Riemannian Hamiltonian as in (17). Then it follows from (12) and (16) that

$$(19) \quad \{h, u_i\} = 0, \quad \forall 1 \leq i \leq s.$$

and from (13) it follows that

$$(20) \quad \{u_i, u_j\} = 0, \quad \forall 1 \leq i, j \leq s,$$

where $\{, \}$ is the Poisson bracket. In other words, $u_i (1 \leq i \leq s)$ are first integrals in involution of the Hamiltonian system $e^{\vec{h}}$.

As before, let Π be the “vertical” distribution, i.e. $\Pi_\lambda = T_\lambda T_{\pi(\lambda)}^*M$, where $\pi : T^*M \rightarrow M$ is the canonical projection. Now we can apply the reduction to the dynamical Lagrangian distributions (\vec{h}, Π) after the first integrals $u_i (1 \leq i \leq s)$ (c.f. Example 5). For this, fix constants c_0, c_1, \dots, c_s , where $c_0 > 0$ is sufficient large. Take a common level set

$$\mathcal{H}_c := \{h = c_0\} \cap \{u_i = c_i, 1 \leq i \leq s\}.$$

Then

$$W_\lambda^c = T_\lambda \mathcal{H}_c / \text{span}\{\vec{h}(\lambda), \vec{u}_i(\lambda), 1 \leq i \leq s\}$$

is a linear symplectic space with the symplectic form σ^c naturally inherited from the symplectic form σ . Moreover,

$$\Pi_\lambda^c = (T_\lambda \mathcal{H}_c \cap \Pi_\lambda) / \text{span}\{\vec{h}(\lambda), \vec{u}_i(\lambda), 1 \leq i \leq s\}$$

is a Lagrangian subspace in W_λ^c . Hence, we get the reduced dynamical Lagrangian distribution (\vec{h}, Π^c) in the linear symplectic space W^c .

Remark 1. The reduction procedure above also applies for the case that the symmetries $\{X_i : 1 \leq i \leq s\}$ are not commutative but still satisfy that $\mathfrak{g} = \text{span}_{\mathbb{R}}\{X_i, 1 \leq i \leq s\}$ is a Lie algebra and the derived Lie algebra $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}]$ is a proper Lie subalgebra. Indeed, take a basis of $\mathfrak{g}^2 : (g_1, \dots, g_k)$ and complete it to a basis of $\mathfrak{g} : (g_1, \dots, g_k, g_{k+1}, \dots, g_s)$. Then if we select a level set $c = (c_1, \dots, c_s)$ such that

$$(21) \quad c_i = 0, \quad 1 \leq i \leq k,$$

then one can see that \mathfrak{g} is commutative on this level set. Therefore, it reduces to commutative case and then the Poisson reduction can be applied. Actually, it is equivalent to considering a sub-Riemannian structure on the manifold obtained by reduction of the original one by g_1, \dots, g_k on which the symmetries consist of a commutative Lie algebra $\mathfrak{g}/\mathfrak{g}^2$.

The (reduced) curvature maps (forms) of (\vec{h}, Π^c) is naturally related to the ambient sub-Riemannian structures (with symmetries), while the later can be reduced to a Riemannian manifold equipped with a \mathbb{R}^s -valued magnetic field. Denote by \widetilde{M} the quotient of M by the leaves of the integral manifold of the involutive distribution spanned by X_1, \dots, X_s and denote the factorization map by $\text{pr} : M \rightarrow \widetilde{M}$. Then \widetilde{M} is (at least locally) a Riemannian manifold equipped with the Riemannian metric g induced from the sub-Riemannian metric. Furthermore, let $\omega = (\omega_i)_{1 \leq i \leq s}$ be the \mathbb{R}^s -valued 1-form defined by $\omega_i|_{\mathcal{D}} = 0$ and $\omega_i(X_j) = \delta_{ij}$, $\forall 1 \leq i, j \leq s$. Then $d\omega = (d\omega_i)_{1 \leq i \leq s}$ induces a \mathbb{R}^s -valued 2-form on \widetilde{M} (still denoted by $d\omega = (d\omega_i)$) and one can define a \mathbb{R}^s -valued tensor $J = (J_i(\tilde{q}), \tilde{q} \in \widetilde{M})$ of type $(1, 1)$ on \widetilde{M} satisfying

$$g_{\tilde{q}}(J_i(\tilde{q})v, w) = d\omega_i(\tilde{q})(v, w), \quad v, w \in T_{\tilde{q}}\widetilde{M}, \tilde{q} \in \widetilde{M}, \quad \forall 1 \leq i \leq s.$$

Let Ξ^c be the s -foliation such that its leaves are integral curves of $\{\vec{u}_i, 1 \leq i \leq s\}$. Let $\text{PR}^c : T^*M \rightarrow T^*M/\Xi^c$ be the canonical projection to the quotient manifold. Now we show that the quotient manifold $N^c = \{u_i = c_i, 1 \leq i \leq s\}/\Xi^c$ can be naturally identified with $T^*\widetilde{M}$. Indeed, a point $\tilde{\lambda}$ in $\{u_i = c_i, 1 \leq i \leq s\}/\Xi^c$ can be identified with a leaf $(\text{PR}^c)^{-1}(\tilde{\lambda})$ of Ξ^c which has a form

$$((e^{-\sum_{i=1}^s t_i X_i})^* p, e^{\sum_{i=1}^s t_i X_i} q),$$

where $\lambda = (p, q) \in (\text{PR}^c)^{-1}(\tilde{\lambda})$, $q \in M$ and $p \in T_q^*M$. On the other hand, any element in $T^*\widetilde{M}$ can be identified with a one-parametric family of pairs $((e^{-\sum_{i=1}^s t_i X_i})^*(p|_{\mathcal{D}}), e^{\sum_{i=1}^s t_i X_i} q)$. The mapping $I^c : \{u_i = c_i, 1 \leq i \leq s\}/\Xi^c \rightarrow T^*\widetilde{M}$ defined by

$$I^c : (e^{-\sum_{i=1}^s t_i X_i})^* p, e^{\sum_{i=1}^s t_i X_i} q \mapsto (e^{-\sum_{i=1}^s t_i X_i})^*(p|_{\mathcal{D}}), e^{\sum_{i=1}^s t_i X_i} q$$

is one-to-one ($p(X_i) = \text{const.}$ is already prescribed) and it defines the required identification.

Before the statement of the main result of the note, let us introduce some notations. Let D^\perp be as in (18). Denote $\mathcal{D}_q^\perp = D^\perp \cap T_q^*M$. Then one has the following series of natural identifications:

$$(22) \quad \Pi_\lambda^c \sim T_q^*M/\mathcal{D}_q^\perp \sim \mathcal{D}_q^* \overset{(\cdot, \cdot)}{\sim} \mathcal{D}_q \sim T_{\text{pr}(q)}\widetilde{M}$$

where $\mathcal{D}_q^* \subseteq T_q^*M$ is the dual space of \mathcal{D}_q . Given $v \in T_\lambda T_q^*M (\sim T_q^*M)$, where $q = \pi(\lambda)$, we can assign a unique vector $v^h \in T_{\text{pr}(q)}\widetilde{M}$ to its equivalence class in $T_q^*M/\mathcal{D}_q^\perp$ by using the identifications (22). Conversely, to any $X \in T_{\text{pr}(q)}\widetilde{M}$ one can assign an equivalence class of $T_\lambda(T_q^*M)/\mathcal{D}_q^\perp$. Denote by $X^v \in T_\lambda T_q^*M$ the unique representative of this equivalence class such that $du_i(X^v) = 0, \forall 1 \leq i \leq s$.

For simplicity, we henceforth denote: $J^c = \sum_{i=1}^s c_i J_i$.

Theorem 2. *The curvature forms r_λ^c of the dynamical Lagrangian distribution (\vec{h}, Π^c) is expressed as follows. For any $v \in \Pi_\lambda^c$,*

$$\begin{aligned} r_\lambda^c(v) &= g(R^\nabla(p^h, v^h)p^h, v^h) + g(\nabla J^c(p^h, v^h), v^h) + \frac{1}{4}g(J^c v^h, J^c v^h) \\ &+ \frac{3}{8(c_0 + W)} (g(J^c p^h, v^h))^2 + \frac{3}{2(c_0 + W)} g(v^h, \nabla W) g(J^c p^h, v^h) \\ &+ \frac{3}{2(c_0 + W)} (g(v^h, \nabla W))^2 + \text{Hess } W(v^h, v^h). \end{aligned}$$

It follows from relations (16), (19) and (20) that $e^{t\vec{h}}$ induces a (reduced) Hamiltonian flow Φ_t on N^c , where $N^c = \{u_i = c_i, 1 \leq i \leq s\}/\Xi^c$, as before. Then following theorem is a direct consequence Theorem 1.

Theorem 3. *Assume that $K^c \subset N^c$ is a compact invariant set of the flow Φ_t on N^c . If the curvature form r_λ^c is negative at every point of K^c , then K^c is a hyperbolic set of the flow Φ_t on N^c .*

As mentioned, the manifold N^c is naturally identified T^*M . Now denote by $S_1\widetilde{M}$ the unit tangent bundle. Combining the previous theorem with Theorem 2, we get the following

Theorem 4. Assume that the reduced Riemannian manifold (\widetilde{M}, g) is compact and has sectional curvature bounded from above by k_{\max} . If the constants c_0, c_1, \dots, c_s satisfy

$$\begin{aligned} & \max_{v, w \in S_1 \widetilde{M}, v \perp w} g(v, \nabla J^c(w; v)) + \frac{1}{4} g(J^c v, J^c v) + \frac{3}{8(c_0 + W)} g(w, J^c v) g(w, J^c v) \\ & + \frac{3}{2(c_0 + W)} g(v, \nabla W) g(J^c w, v) + 3 \left(\frac{\|\nabla W\|}{2(c_0 + W)} \right)^2 + \frac{\|\text{Hess } W\|}{2(c_0 + W)} < -k_{\max}, \end{aligned}$$

then the flow Φ_t is an Anosov flow.

Corollary 1. Pure potential flows, i.e. $J_i = 0 (1 \leq i \leq s)$,

$$(23) \quad \max_{\vec{q} \in \widetilde{M}} \left(3 \left(\frac{\|\nabla W\|}{2(c_0 + W)} \right)^2 + \frac{\|\text{Hess } W\|}{2(c_0 + W)} \right) < -k_{\max}.$$

Corollary 2. Pure magnetic flows, i.e. $W = 0$,

$$(24) \quad \max_{v, w \in S_1 \widetilde{M}, v \perp w} cg(v, \nabla J(w; v)) + c^2 g(Jv, Jv) < -k_{\max}.$$

Remark 2. The left-hand side of the inequality in Theorem 4 is always positive, because the second term inside the max is positive and the first term can be made nonnegative, if necessary, by changing the sign of w . Hence, Theorem 4 makes sense only if $k_{\max} < 0$.

The flow Φ_t on N^c can be considered as a perturbation of the Riemannian geodesic flow: the flow Φ_t on N^c remains to be an Anosov flow for sufficient small constants $c_i (1 \leq i \leq s)$ and for proper potential function W (sufficient small norm of W and its derivatives). When $s = 1$, it coincides with Theorem 4.1 (the case of Gaussian thermostats of external fields $E = 0$ there) in [13]; when $s = 1$ and $W = 0$, the condition of Anosov magnetic flows (24) coincides with the main results in [8].

3. PROOF OF THE MAIN RESULTS

The rest of the note is devoted to the proof of Theorem 2.

3.1. Reduced curvature maps. As before, fix constants c_0, c_1, \dots, c_s , where $c_0 > 0$ is sufficient large. Let $J_\lambda^c(t)$ be the Jacobi curves associated with the reduced dynamical Lagrangian distribution (\vec{h}, Π^c) , namely,

$$(25) \quad J_\lambda^c(t) := e_*^{-t\vec{h}} \Pi_{e^{t\vec{h}}\lambda}^c.$$

Lemma 1. The reduced Jacobi curve $J_\lambda^c(\cdot)$ is a regular monotone nondecreasing curve in Lagrange Grassmannian $L(W_\lambda^c)$.

Proof. First note that if $\bar{\lambda} = e^{\vec{t}\vec{h}}\lambda$ and $\phi : W_\lambda^c \rightarrow W_{\bar{\lambda}}^c$ is a symplectic transformation induced in the natural way by a linear mapping $e_*^{\vec{t}\vec{h}} : T_\lambda \mathcal{H}_c \rightarrow T_{\bar{\lambda}} \mathcal{H}_c$, where, as before,

$$\mathcal{H}_c := \{h = c_0\} \cap \{u_i = c_i, 1 \leq i \leq s\}.$$

Then by (25) we have

$$(26) \quad J_\lambda^c(t) = \phi(J_{\bar{\lambda}}^c(t - \vec{t})).$$

Further, it turns out (see, for example, [2, Proposition 1]) that the velocity of the Jacobi curve $J_\lambda^c(\cdot)$ at $t = 0$ is equal to the restriction of the Hessian of h to the tangent space to Π_λ^c at the point λ . This together with the relation (26) and the construction of Π_λ^c implies easily that $J_\lambda^c(t)$ is a regular monotone nondecreasing curve. \square

Theorem 5. Let $\Lambda(\cdot)$ be a regular curve in the Lagrange Grassmannian $L(G)$ of a $2n$ -dimensional linear symplectic space G . Then there exists a moving Darboux frame $(E(t), F(t))$ of G :

$$E(t) = (e_1(t), \dots, e_n(t)), \quad F(t) = (f_1(t), \dots, f_n(t))$$

such that $\Lambda(t) = \text{span}\{E(t)\}$ and there exists a one-parametric family of linear self-adjoint operators $\mathfrak{R}(t) : \Lambda(t) \rightarrow \Lambda(t)$ satisfying

$$(27) \quad \begin{cases} E'(t) = F(t), \\ F'(t) = -R(t)E(t). \end{cases}$$

The moving frame $(E(t), F(t))$ is called a normal moving frame of $\Lambda(t)$ and the linear operator $\mathfrak{R}(t)$ is called the curvature map of $\Lambda(t)$. A moving frame $(\tilde{E}(t), \tilde{F}(t))$ is a normal moving frame of $\Lambda(t)$ if and only there exists a constant orthogonal matrix U of size $n \times n$ such that

$$(28) \quad \tilde{E}(t) = E(t)U, \quad \tilde{F}(t) = F(t)U.$$

Remark 3. Note that from (27) it follows that if $(\tilde{E}(t), \tilde{F}(t))$ is a Darboux moving frame such that $\tilde{E}(t)$ is an orthonormal frame of $\Lambda(t)$ and $\text{span}\{\tilde{F}(t)\} = \Lambda^{\text{trans}}(t)$. Then there exists a curve of antisymmetric matrices $B(t)$ such that

$$(29) \quad \begin{cases} \tilde{E}'(t) = \tilde{E}(t)B(t) + \tilde{F}_a(t) \\ \tilde{F}'(t) = -\tilde{E}(t)\tilde{\mathcal{R}}(t) + \tilde{F}(t)B(t), \end{cases}$$

where $\tilde{\mathcal{R}}(t)$ is the matrix of the curvature map $\mathfrak{R}(t)$ on $\Lambda(t)$ w.r.t. the basis $\tilde{E}(t)$.

As a matter of fact, normal moving frames define a principal $O(n)$ -bundle of symplectic frame in G endowed with a canonical connection. Also, relations (28) imply that the following n -dimensional subspaces

$$(30) \quad \Lambda^{\text{trans}}(t) = \text{span}\{F(t)\}$$

of G does not depend on the choice of the normal moving frame. It is called the *canonical complement* of $\Lambda(t)$ in G . Moreover, the subspaces $\Lambda(t)$ and $\Lambda^{\text{trans}}(t)$ are endowed with the *canonical Euclidean structure* such that the tuple of vectors $E(t)$ and $F(t)$ constitute an orthonormal frame w.r.t. to it, respectively.

Finally, the linear map from $\Lambda(t)$ to $\Lambda(t)$ with the matrix $R(t)$ from (27) in the basis $\{E(t)\}$, is independent of the choice of normal moving frames. It will be denoted by $\mathfrak{R}(t)$ and it is called the *curvature map* of the curve $\Lambda(t)$.

Now we apply the above results for curves in Lagrange Grassmannians to sub-Riemannian structures. Since $\mathfrak{J}_\lambda^c(0)$ and Π_λ^c can be naturally identified, there is a canonical splitting of W_λ^c :

$$(31) \quad W_\lambda^c = \Pi_\lambda^c \oplus \tilde{\mathfrak{J}}^c(\lambda),$$

where $\tilde{\mathfrak{J}}^c(\lambda) = \text{span}(F^\lambda(0))$ is the canonical complement. In other words, $\tilde{\mathfrak{J}}^c(\lambda)$ is actually a (nonlinear) Ehresmann connection of Π_λ^c in W_λ^c . It also follows that the subspaces Π_λ^c and $\tilde{\mathfrak{J}}^c(\lambda)$ are equipped with a canonical Euclidean structure. Moreover, one can define the curvature map of the dynamical Lagrangian distribution (\vec{h}, Π^c) , i.e. $\mathfrak{R}_\lambda^c : \Pi_\lambda^c \rightarrow \Pi_\lambda^c$ such that $\mathfrak{R}_\lambda^c = \mathfrak{R}_\lambda(0)$ of the curvature maps of the Jacobi curve $\mathfrak{J}_\lambda^c(\cdot)$ at $t = 0$. This curvature maps are intrinsically related to the sub-Riemannian structure and will be called the *reduced curvature map* of the sub-Riemannian structure.

Let $\lambda \in T^*M$ and let $\lambda(t) = e^{t\vec{h}}\lambda$. Assume that $(E^\lambda(t), F^\lambda(t))$ is a normal moving frame of the Jacobi curve $\mathfrak{J}_\lambda^c(t)$ attached at point λ . Let \mathfrak{E} be the Euler field on T^*M , i.e. the infinitesimal generator of the homotheties on its fibers. Clearly $T_\lambda(T^*M) = T_\lambda\mathcal{H}_c \oplus \mathbb{R}\mathfrak{E}(\lambda) \oplus \text{span}\{\partial_{u_i}(\lambda), 1 \leq i \leq s\}$. The flow $e^{t\vec{h}}$ on T^*M induces the push-forward maps $e_*^{t\vec{h}}$ between the corresponding tangent spaces $T_\lambda T^*M$ and $T_{\lambda(t)} T^*M$, which in turn induce naturally the maps between the spaces $T_\lambda(T^*M)/\text{span}\{\vec{h}(\lambda), \vec{u}_i(\lambda), 1 \leq i \leq s\}$ and $T_{\lambda(t)} T^*M/\text{span}\{\vec{h}(\lambda(t)), \vec{u}_i(\lambda(t)), 1 \leq i \leq s\}$. The map \mathcal{K}^t between $T_\lambda(T^*M)/\text{span}\{\vec{h}(\lambda), \vec{u}_i(\lambda), 1 \leq i \leq s\}$ and $T_{\lambda(t)} T^*M/\text{span}\{\vec{h}(\lambda(t)), \vec{u}_i(\lambda(t)), 1 \leq i \leq s\}$, sending $E^\lambda(0)$ to $e_*^{t\vec{h}} E^\lambda(t)$, $F^\lambda(0)$ to $e_*^{t\vec{h}} F^\lambda(t)$, and the equivalence class of $\mathfrak{E}(\lambda), \partial_{u_i}(\lambda) (1 \leq i \leq s)$ to the equivalence class of $\mathfrak{E}(e^{t\vec{h}}\lambda), \partial_{u_i}(\lambda(t)) (1 \leq i \leq s)$, is independent of the choice of normal moving frames. The map \mathcal{K}^t is called the *parallel transport* along the extremal $e^{t\vec{h}}\lambda$ at time t . For any $v \in T_\lambda(T^*M)/\text{span}\{\vec{h}(\lambda), \vec{u}_i(\lambda), 1 \leq i \leq s\}$, its image $v(t) = \mathcal{K}^t(v)$ is called the *parallel transport of v at time t* . Note that from the definition of the reduced Jacobi curves and the construction of normal moving frames it follows that the restriction of the parallel transport \mathcal{K}_t

to the vertical subspace $T_\lambda(T_{\pi(\lambda)}^*M)$ of $T_\lambda(T^*M)$ can be considered as a map onto the vertical subspace $T_{\lambda(t)}(T_{\pi(\lambda(t))}^*M)$ of $T_{\lambda(t)}(T^*M)$. A vertical vector field V is called *parallel* if $V(e^{t\vec{h}}\lambda) = \mathcal{K}^t(V(\lambda))$.

Example 6 (Riemannian geodesic flow) In this case, $\mathcal{D} = TM, W = 0$ and there is no symmetries at all ($s = 0$). In [3] the reduced curvature map was expressed by the Riemannian curvature tensor. If we adopt the notations from Example 3 and take the constants $c_0 = \frac{1}{2}, c_i = 0 (1 \leq i \leq s)$. Then

$$(32) \quad \mathfrak{R}_\lambda^c(v) = R^\nabla(p^h, v^h)p^h, \quad \forall \lambda = (q, p) \in \mathcal{H}_c, q \in M, p \in T_q^*M, v \in \Pi_\lambda^c.$$

Given a vector $X \in T_qM$ denote by ∇_X its lift to the Levi-Civita connection, considered as an Ehresmann connection on T^*M . Then by constructions the Hamiltonian vector field \vec{h} is horizontal and satisfies $\vec{h} = \nabla_{p^h}$. Take any $v, w \in \Pi_\lambda^c$ and let V be a vertical vector field such that $V(\lambda) = v$. From (32), structure equation (27), and the fact that the Levi-Civita connection (as an Ehresmann connection on T^*M) is a Lagrangian distribution (c.f. [3]) it follows that the Riemannian curvature tensor satisfies the following identity:

$$(33) \quad \langle R^\nabla(p^h, v^h)p^h, w^h \rangle = -\sigma([\nabla_{p^h}, \nabla_{V^h}](\lambda), \nabla_{w^h}). \quad \square$$

3.2. Proof of Theorem 2. We first express the canonical complement in terms of the Levi-Civita connection of the Riemannian metric and the tensor J_i^c and then we can give the proof of Theorem 2 using some calculus formulae which is developed in [10].

3.2.1. The canonical complement $\tilde{\mathfrak{J}}^c(\lambda)$. The restriction of the parallel transport \mathcal{K}^t to Π_λ^c is characterized by the following two properties:

- (1) \mathcal{K}^t is an orthogonal transformation of spaces Π_λ^c and $\Pi_{e^{t\vec{h}}\lambda}^c$;
- (2) The space $\text{span}\{\frac{d}{dt}((e^{-t\vec{h}})_*(\mathcal{K}^t v))|_{t=0} : v \in \Pi_\lambda^c\}$ is isotropic.

Then $\tilde{\mathfrak{J}}^c(\lambda) = \text{span}\{\frac{d}{dt}((e^{-t\vec{h}})_*(\mathcal{K}^t v))|_{t=0} : v \in \Pi_\lambda^c\}$.

To express $\tilde{\mathfrak{J}}^c(\lambda)$ in terms of the Riemannian manifold and the magnetic field, we show the decomposition of the symplectic form σ (the standard symplectic form on $T^*\widetilde{M}$) and the Hamiltonian field \vec{h} . One can see that the diffeomorphism I^c , defined as before, are not in general symplectic. Indeed, each level set inherits a symplectic structure depending on the choice of the level $\{c_i : 1 \leq i \leq s\}$.

By the construction of the map I^c , for any vector field X on $T^*\widetilde{M}$, we can assign the vector field \underline{X} on T^*M s.t. $PR_*^c \underline{X} = ((I^c)^{-1})_* X$ and $\pi_* \underline{X} \in \mathcal{D}$. In the following, denote by ∇_{p^h} the lift of p^h to $T^*\widetilde{M}$ with respect to the Levi-Civita connection and denote $\Omega^c = \sum_{i=1}^s c_i d\omega_i$ and $\bar{\sigma} = (I^c \circ PR^c)^* \sigma$. We will denote by $\tilde{\sigma}$ the standard symplectic form on $T^*\widetilde{M}$. The proof of the following lemma is complete similar to that of Lemma 3.1-3.3 in [10] and thus is omitted.

Lemma 2. *The following decomposition formulae hold.*

- (1) On the level set $\{u_i = c_i, i = 1, \dots, s\}$, $\sigma = \bar{\sigma} - (\pi \circ \text{pr})^*(\Omega^c)$;
- (2) For any vectors $X, V \in T_\lambda T^*\widetilde{M}$ with $\pi_* V = 0$ we have $\sigma(X, v) = g(\pi_* X, V^h)$;
- (3) $\vec{h}(p, q) = \underline{\nabla_{p^h}} - (J^c p^h)^v + \vec{W}$.

Comparing with the sub-Riemannian geodesic problem, we will develop some additional calculus formulae about the potential W .

Lemma 3. *Let V_1, V_2 be the vector fields on T^*M with $\pi_* V_1 = \pi_* V_2 = 0$. Then*

- (1) $\vec{W} = -(\nabla W)^v$;
- (2) $\bar{\sigma}([\vec{W}, \underline{\nabla_{V_1^h}}], \underline{\nabla_{V_2^h}}] = -\text{Hess } W(V_1^h, V_2^h)$;
- (3) $\vec{W}(g(V_1^h, V_2^h)) - g([\vec{W}, V_1]^h, V_2^h) - g(V_1^h, [\vec{W}, V_2]^h) = 0$.

Proof. (1) From the second item of the last lemma it follows that

$$\sigma(\vec{W}, V_1^h) = V_1^h(W) = dW(V_1^h) = g(\nabla W, V_1^h) = -\sigma((\nabla W)^v, V_1^h).$$

Taking into account that $\pi_* \vec{W} = 0$, we get

$$\vec{W} = -(\nabla W)^v, \quad \text{span}\{\partial_{u_i}, i = 1, \dots, s\}.$$

On the other hand, it follows from (50) and item (1) of the present lemma that

$$\sigma(\vec{u}_i, \vec{W}) = -\vec{u}_i(W) = -X_i(W) = 0, \quad i = 1, \dots, s.$$

Thus, we get the required identity $\vec{W} = -(\nabla W)^v$.

(2) Both sides of the required identity are linear w.r.t. V_1, V_2 , respectively, thus it is sufficient to prove it for the case that V_1^h, V_2^h are both vector fields on $T^*\widetilde{M}$. But for this case the required identity is a direct consequence of the definition of the Hessian.

(3) Left-hand side is linear w.r.t. V_1, V_2 , respectively, thus it is sufficient to prove it for the case that V_1^h, V_2^h are both vector fields on $T^*\widetilde{M}$. In this case, the vector fields $(V_1^h)^v, (V_2^h)^v$, together with $(\nabla W)^v$ are all constants on the fibers of T^*M and then the required identity become trivial. \square

Given any $X \in \Pi_\lambda^c$ denote by $\widetilde{\nabla}_{X^h}$ the lift of X to $\widetilde{\mathfrak{J}}^c(\lambda)$: the unique vector $\widetilde{\nabla}_{X^h} \in \widetilde{\mathfrak{J}}^c(\lambda)$ such that $(\text{pr} \circ \pi)_* \widetilde{\nabla}_{X^h} = X^h$. Then there exist the unique $B \in \text{End}(\Pi_\lambda^c)$ and $\tilde{A} \in (\Pi_\lambda^c)^*$ such that

$$(34) \quad \widetilde{\nabla}_{v^h} = \underline{\nabla}_{v^h} + Bv, \quad \forall v \in \Pi_\lambda^c,$$

where ∇ stands for the lifts to the Levi-Civita connection on $T^*\widetilde{M}$, as before.

Lemma 4. *The linear operator B is antisymmetric w.r.t. the canonical Euclidean structure in Π_λ^c .*

Proof. Fix a point $\bar{\lambda} \in T^*M$ and consider a small neighborhood U of $\bar{\lambda}$. Let $\mathcal{E} = \{\mathcal{E}^i\}_{i=1}^{m-1}$ be a frame of Π_λ^c (i.e. $\Pi_\lambda^c = \text{span } \mathcal{E}(\lambda)$) for any $\lambda \in U$ such that the following four conditions hold

- (1) \mathcal{E} is orthogonal w.r.t. the canonical Euclidean structure on Π_λ^c ;
- (2) Each vector field \mathcal{E}^i is parallel w.r.t the canonical parallel transport \mathcal{K}_t , i.e. $\mathcal{E}^i(e^t \vec{h} \lambda) = \mathcal{K}^t \mathcal{E}^i(\lambda)$ for any λ and t such that $\lambda, e^t \vec{h} \lambda \in U$;
- (3) The vector fields $(J^c p^h)^v$ and \mathcal{E}^i commute on $U \cap T_{\pi(\bar{\lambda})}^* M$;
- (4) The vector fields \vec{u}_i , $\forall 1 \leq i \leq s$ and \mathcal{E}^i commute on $U \cap T_{\pi(\bar{\lambda})}^* M$.

Note that the frame \mathcal{E} with properties above exists, because the Hamiltonian vector field \vec{h} is transversal to the fibers of T^*M and it commutes with \vec{u}_i , $\forall 1 \leq i \leq s$.

From the property (2) of the parallel transport \mathcal{K}^t in this subsection it follows that

$$(35) \quad \widetilde{\nabla}_{(\mathcal{E}^i)^h} = -\text{ad} \vec{h} \mathcal{E}^i$$

Using the above defined identification $I^c : N^c \rightarrow T^*\widetilde{M}$, one can look on the restriction of the tuple of vector fields \mathcal{E} to the submanifold $\{u_i = c_i, i = 1, \dots, s\}$ as on the tuple of the vertical vector fields of $T^*\widetilde{M}$ (which actually span the tangent to the intersection of the fiber of $T^*\widetilde{M}$ with the level to the corresponding Riemannian Hamiltonian). Then first the tuple \mathcal{E} is the tuple of orthonormal vector fields (w.r.t. the canonical Euclidean structure on the fibers of $T^*\widetilde{M}$, induced by the Riemannian metric g). Further, by the equations (29) the Levi-Civita connection of g is characterized by the fact that there exists a field of antisymmetric operators $\widetilde{B} \in \text{End}(\Pi_\lambda^c)$ such that

$$(36) \quad [\nabla_{p^h}, \widetilde{\mathcal{E}}^i(\lambda)] = -\nabla_{(\widetilde{\mathcal{E}}^i(\lambda))^h} - \widetilde{B} \widetilde{\mathcal{E}}^i(\lambda)$$

On the other hand, from (35), (36), using the second item of Lemma 2 and the property (3) of \mathcal{E}^i , one has

$$(37) \quad \widetilde{\nabla}_{(\mathcal{E}^i)^h} = -\text{ad} \vec{h} \mathcal{E}^i = -[\underline{\nabla}_{p^h} - \sum_{i=1}^s c_i (J_i p^h)^v + \vec{W}, \mathcal{E}^i] = \underline{\nabla}_{(\mathcal{E}^i(\lambda))^h} + \widetilde{B} \mathcal{E}^i(\lambda) + [\vec{W}, \mathcal{E}^i(\lambda)].$$

From item (3) of Lemma 3 and property (1) of \mathcal{E}^i it follows

$$(38) \quad g([\vec{W}, \mathcal{E}^i]^h, (\mathcal{E}^j)^h) + g((\mathcal{E}^i)^h, ([\vec{W}, \mathcal{E}^j]^h)) = \vec{W} (g((\mathcal{E}^i)^h, (\mathcal{E}^j)^h)) = 0.$$

Therefore, from (37) and (38) we conclude that B is antisymmetric. \square

Lemma 5. *The operator B satisfies*

$$(39) \quad (Bv)^h = -\frac{1}{2} J^c v^h \mod p^h, \quad \forall v \in \Pi_\lambda^c.$$

Proof. Since $\tilde{\mathfrak{J}}^c(\lambda)$ is an isotropic subspace, we have

$$\sigma(\tilde{\nabla}_{v_1^h}, \tilde{\nabla}_{v_2^h}) = 0, \quad \forall v_1, v_2 \in \Pi_\lambda^c.$$

On the other hand, using Proposition 2, the fact that the Levi-Civita connection (as an Ehresmann connection) is a Lagrangian distribution in $T^*\tilde{M}$ and Lemma 2, we get

$$\begin{aligned} 0 = \sigma(\tilde{\nabla}_{v_1^h}, \tilde{\nabla}_{v_2^h}^c) &= \left((I^c \circ \text{PR}^c)^* \tilde{\sigma} - (\text{pr} \circ \pi)^* \Omega^c \right) (\nabla_{v_1^h} + Bv_1, \nabla_{v_2^h} + Bv_2) \\ &= -\Omega^c(v_1^h, v_2^h) - g((Bv_1)^h, v_2^h) + g((Bv_2)^h, v_1^h) \\ &= -g(J^c v_1^h, v_2^h) - g((Bv_1)^h, v_2^h) + g((B^* v_1)^h, v_2^h). \end{aligned}$$

where B^* is the dual of B w.r.t. the Euclidean structure in Π_λ^c . Taking into account that B is antisymmetric, we get

$$(40) \quad (Bv)^h = -\frac{1}{2} J^c v^h \mod p^h.$$

□

Corollary 3. *The canonical complement $\tilde{\mathfrak{J}}^c(\lambda)$ can be expressed as follows:*

$$\tilde{\mathfrak{J}}^c(\lambda) = \left\{ \underline{\nabla}_{v^h} - \frac{1}{2} (J^c v^h)^v - \frac{1}{2 \|p^h\|^2} \cdot g(v^h, J^c p^h + 2\nabla W) (p^h)^v, \quad v \in \Pi_\lambda^c \right\}.$$

Proof. It follows from (34) and (39) that there exist $A \in (\Pi_\lambda^c)^*$ such that

$$\tilde{\nabla}_{v^h} = \underline{\nabla}_{v^h} - \frac{1}{2} (J^c v^h)^v + A(v)(p^h)^v.$$

Note that $\sigma(\vec{h}, (p^h)^v) = g(p^h, p^h) = \|p^h\|^2$. Hence, from the fact that $\tilde{\nabla}_{v^h}$ is tangent to the Hamiltonian vector field \vec{h} , we get easily that

$$A(v) = -\frac{1}{2 \|p^h\|^2} \cdot g(v^h, J^c p^h + 2\nabla W),$$

which completes the proof of the corollary. □

3.2.2. The reduced curvature map. As a direct consequence of structure equation (27), we get the following preliminary descriptions of the reduced curvature map:

Proposition 1. *Let $v \in \Pi_\lambda^c$. Let V be a parallel vector field such that $V(\lambda) = v$. Then the curvature maps satisfy the following identities:*

$$(41) \quad g((\mathfrak{R}_\lambda^c v)^h, v^h) = -\sigma(\text{ad} \vec{h}(\tilde{\nabla}_{V^h}), \tilde{\nabla}_{v^h}).$$

It follows that in order to calculate the reduced curvature map it is sufficient to know how to express the Lie bracket of vector fields on the cotangent bundle T^*M via the covariant derivatives of Levi-Civita connection on $T^*\tilde{M}$.

Proposition 2. *For any tensors A, B of type $(1, 1)$ on \tilde{M} , the following identity holds:*

- (1) $[(Ap^h)^v, (Bp^h)^v] = (B(Ap^h))^v - (A(Bp^h))^v,$
- (2) $[\underline{\nabla}_{p^h}, (Ap^h)^v] = -\underline{\nabla}_{Ap^h} + ((\nabla_{p^h} A)p^h)^v.$

For simplicity, denote $\bar{\sigma} = (I^c \circ \text{PR}^c)^* \sigma$ and $\Omega^c = \sum_{i=1}^s c_i d\omega_i$, as before. As in the proof of Lemma 4, we can take a parallel vector field V such that $V(\lambda) = v$ and

$$(42) \quad [(J^c p^h)^v, V](\bar{\lambda}) = 0, \quad \bar{\lambda} \in U \cap T_q^* M,$$

where U is a neighborhood of λ . Similar to Proposition 4 of [10], we have

Lemma 6. *Let V, V_1, V_2 be vector fields on T^*M with $\pi_* V = \pi_* V_1 = \pi_* V_2 = 0$. Then*

- (1) $[(J^c p^h)^v, (J^c V^h)^v]^h = J^c([(J^c p^h)^v, (V^h)^v])^h,$
- (2) $\bar{\sigma}([(J^c p^h)^v, \underline{\nabla}_{V_1^h}], \underline{\nabla}_{V_2^h}) = g(\nabla J^c(p^h, V_1^h), V_2^h),$
- (3) $(\text{pr} \circ \pi)_*([(J^c p^h)^v, \underline{\nabla}_{V^h}]) = J^c V^h,$
- (4) $(\text{pr} \circ \pi)_*([\underline{\nabla}_{p^h}, \underline{\nabla}_{V^h}]) = \frac{1}{2} J^c V^h - \frac{1}{2 \|p^h\|^2} g(J^c V^h, p^h) p^h.$

Let us simplify the right-hand side of the identity (41). First, from the last line of the structural equations (27) it follows that

$$(43) \quad (\text{pr} \circ \pi)_*(\text{ad}\vec{h}(\widetilde{\nabla}_{V^h})) \in \mathbb{R}p^h.$$

Besides,

$$\sigma\left(\vec{h}, \frac{1}{2}(J^c v^h)^v + \frac{1}{2\|p^h\|^2}g(v^h, J^c p^h)(p^h)^v\right) = \frac{1}{2}g(p^h, J^c v^h) + \frac{1}{2\|p^h\|^2}g(v^h, J^c p^h)g(p^h, p^h) = 0.$$

Hence from Lemma 2 and Corollary 3 it follows that

$$\begin{aligned} & \sigma(\text{ad}\vec{h}(\nabla_{V^h}), \nabla_{v^h}) = \sigma\left(\text{ad}\vec{h}(\nabla_{V^h}), \underline{\nabla_{v^h}} - \frac{1}{\|p^h\|^2}g(v^h, \nabla W)(p^h)^v\right) \\ &= \sigma\left(\underline{[\nabla_{p^h} - (J^c p^h)^v, \underline{\nabla_{V^h}}]} - \frac{1}{2}(J^c V^h)^v - \frac{1}{2\|p^h\|^2} \cdot g(V^h, J^c p^h)(p^h)^v, \underline{\nabla_{v^h}}\right) \\ &+ \sigma\left(\underline{[\nabla_{p^h} - (J^c p^h)^v, \underline{\nabla_{V^h}}]} - \frac{1}{2}(J^c V^h)^v - \frac{1}{2\|p^h\|^2} \cdot g(V^h, J^c p^h)(p^h)^v, -\frac{1}{\|p^h\|^2} \cdot g(v^h, \nabla W)(p^h)^v\right) \\ &+ \sigma\left(\underline{[\nabla_{p^h} - (J^c p^h)^v, -\frac{1}{\|p^h\|^2} \cdot g(V^h, \nabla W)(p^h)^v]}, \underline{\nabla_{v^h}} - \frac{1}{\|p^h\|^2}g(v^h, \nabla W)(p^h)^v\right) \\ &+ \sigma\left(\underline{[\vec{W}, \underline{\nabla_{V^h}}]}, \underline{\nabla_{v^h}} - \frac{1}{\|p^h\|^2}g(v^h, \nabla W)(p^h)^v\right) \\ &+ \sigma\left(\underline{[\vec{W}, -\frac{1}{2}(J^c V^h)^v - \frac{1}{2\|p^h\|^2} \cdot g(V^h, J^c p^h + 2\nabla W)(p^h)^v]}, \underline{\nabla_{v^h}}\right) \\ &=: T_1 + T_2 + T_3 + T_4 + T_5 \end{aligned}$$

We will deal with the terms $T_i (1 \leq i \leq 5)$ in steps.

Step 1 It follows identity (33) that

$$(44) \quad \bar{\sigma}(\underline{[\nabla_{p^h}, \underline{\nabla_{V^h}}]}, \underline{\nabla_{v^h}}) = -g(R^\nabla(p^h, v^h)p^h, v^h).$$

Also it follows from item (3) of Lemma 6 and item (4) of Lemma 6 that

$$\begin{aligned} & \Omega^c((\text{pr} \circ \pi)_*(\underline{[\nabla_{p^h}, \underline{\nabla_{V^h}}]}), v^h) + \frac{1}{2}\bar{\sigma}(\underline{[\nabla_{p^h}, (J^c V^h)^v]}, \underline{\nabla_{v^h}}) \\ &= -g((\text{pr} \circ \pi)_*(\underline{[\nabla_{p^h}, \underline{\nabla_{V^h}}]}), J^c v^h) + \frac{1}{2}\bar{\sigma}(\underline{[\nabla_{p^h}, \underline{\nabla_{V^h}}]}, (J^c v^h)^v) \\ &= -\frac{1}{2}g((\text{pr} \circ \pi)_*(\underline{[\nabla_{p^h}, \underline{\nabla_{V^h}}]}), J^c v^h) \\ &= -\frac{1}{4}\|J^c v^h\|^2 + \frac{1}{4\|p^h\|^2}(g(J^c v^h, p^h))^2 \end{aligned}$$

Also it follows from straightforward computations that

$$(45) \quad \Omega^c((\text{pr} \circ \pi)_*(\underline{[\nabla_{p^h}, (J^c v^h)^v]}), v^h) = -\Omega^c(J^c v^h, v^h) = \|J^c v^h\|^2$$

Also it follows from item (2) of Proposition 2 that

$$(46) \quad \bar{\sigma}(\underline{[\nabla_{p^h}, (p^h)^v]}, \underline{\nabla_{v^h}}) = \bar{\sigma}(-\underline{\nabla_{p^h}}, \underline{\nabla_{v^h}}) = 0.$$

and

$$(47) \quad \Omega^c((\text{pr} \circ \pi)_*(\underline{[\nabla_{p^h}, (p^h)^v]}), v^h) = -\Omega^c(p^h, v^h) = g(p^h, J^c v^h).$$

It follows from item (2) of Lemma 6 that

$$(48) \quad \bar{\sigma}((J^c p^h)^v, \underline{\nabla_{V^h}}), \underline{\nabla_{v^h}}) = g(\nabla J^c(p^h, v^h), v^h).$$

Applying item (3) of Lemma 6, we get

$$(49) \quad \Omega^c((\text{pr} \circ \pi)_*(\underline{[(J^c p^h)^v, \underline{\nabla_{V^h}}]}), v^h) = \Omega^c(J^c v^h, v^h) = -\|J^c v^h\|^2.$$

And it follows from (42) and item (1) of Lemma 6 that

$$(50) \quad [(J^c p^h)^v, (J^c V^h)^v] = 0.$$

And it follows from item (1) of Proposition 2 that

$$(51) \quad [(J^c p^h)^v, (p^h)^v] = 0.$$

Then it follows that

$$\sigma \left([-(J^c p^h)^v, -\frac{1}{2\|p^h\|^2} \cdot g(V^h, J^c p^h + 2\nabla W)(p^h)^v], \underline{\nabla_{v^h}} \right) = 0$$

Summarizing all the calculations above, we have

$$(52) \quad T_1 = -g(R^\nabla(p^h, v^h)p^h, v^h) - g(\nabla J^c(p^h, v^h), v^h) - \frac{1}{4}\|J^c v^h\|^2 - \frac{3}{4\|p^h\|^2} (g(p^h, J^c v^h))^2$$

Step 2 Again, it follows from item (4) of Lemma 6 that

$$(53) \quad \sigma([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}], (p^h)^v) = -\frac{1}{2}g(J^c v^h, p^h).$$

And it follows from straightforward computations that

$$(54) \quad \sigma \left([\underline{\nabla_{p^h}}, \frac{1}{2}(J^c V^h)^v + \frac{1}{2\|p^h\|^2} \cdot g(V^h, J^c p^h)(p^h)^v], (p^h)^v \right) = 0.$$

And it follows from item (3) of Lemma 6 that

$$(55) \quad \sigma([(J^c V^h)^v, \underline{\nabla_{V^h}}], (p^h)^v) = g(J^c v^h, p^h).$$

Hence, we have

$$(56) \quad T_2 = -\frac{1}{\|p^h\|^2}g(J^c p^h, v^h)g(\nabla W, v^h).$$

Step 3 It follows from item (2) of Proposition 2 that $[\underline{\nabla_{p^h}}, (p^h)^v] = -\underline{\nabla_{p^h}}$, hence

$$\begin{aligned} & \sigma \left([\underline{\nabla_{p^h}}, -\frac{1}{\|p^h\|^2} \cdot g(V^h, \nabla W)(p^h)^v], \underline{\nabla_{v^h}} - \frac{1}{\|p^h\|^2}g(v^h, \nabla W)(p^h)^v \right) \\ &= -\frac{1}{\|p^h\|^2} \cdot g(v^h, \nabla W) \sigma \left([\underline{\nabla_{p^h}}, (p^h)^v], \underline{\nabla_{v^h}} - \frac{1}{\|p^h\|^2}g(v^h, \nabla W)(p^h)^v \right) \\ &= -\frac{1}{\|p^h\|^2} \cdot g(v^h, \nabla W) \sigma \left(-\underline{\nabla_{p^h}}, \underline{\nabla_{v^h}} - \frac{1}{\|p^h\|^2}g(v^h, \nabla W)(p^h)^v \right) \\ &= -\frac{1}{\|p^h\|^2} \cdot g(v^h, \nabla W)g(J^c p^h, v^h) - \frac{1}{\|p^h\|^2} \cdot (g(V^h, \nabla W))^2. \end{aligned}$$

And it follows from (51) that

$$(57) \quad \sigma \left([(J^c p^h)^v, -\frac{1}{\|p^h\|^2} \cdot g(V^h, \nabla W)(p^h)^v], \underline{\nabla_{v^h}} \right) = 0.$$

Hence,

$$(58) \quad T_3 = -\frac{1}{\|p^h\|^2} \cdot g(v^h, \nabla W)g(J^c p^h, v^h) - \frac{1}{\|p^h\|^2} \cdot (g(V^h, \nabla W))^2$$

Step 4 From item (2) of Lemma 3, we have

$$(59) \quad \bar{\sigma} \left([\vec{W}, \underline{\nabla_{V^h}}], \underline{\nabla_{v^h}} \right) = -\text{Hess } W(v^h, v^h).$$

Lemma 7. *The following identity holds.*

$$(\text{pr} \circ \pi)_*([\vec{W}, \underline{\nabla_{V^h}}]) = \frac{1}{\|p^h\|^2}g(v^h, \nabla W)p^h.$$

Proof. First of all, from (43), item (3) of Lemma 2 and Lemma 3 it follows

$$(\text{pr} \circ \pi)_*([\underline{\nabla_{p^h}} - (J^c p^h)^v + \vec{W}, \underline{\nabla_{V^h}} - \frac{1}{2}(J^c V^h)^v]) = 0, \quad \text{mod } p^h.$$

Then together with item (3)-(4) of Lemma 6 and item (2) of Proposition 2 we have

$$(\text{pr} \circ \pi)_*([\vec{W}, \underline{\nabla_{V^h}}]) = 0, \quad \text{mod } p^h.$$

Furthermore, from the classical Cartan's formula we have

$$(60) \quad 0 = d\sigma(\vec{W}, \underline{\nabla_{V^h}}, (p^h)^v) = \vec{W}(\sigma(\underline{\nabla_{V^h}}, (p^h)^v)) - \underline{\nabla_{V^h}}(\sigma(\vec{W}, (p^h)^v)) + (p^h)^v(\sigma(\vec{W}, \underline{\nabla_{V^h}})) \\ - \sigma([\vec{W}, \underline{\nabla_{V^h}}], (p^h)^v) + \sigma([\vec{W}, (p^h)^v], \underline{\nabla_{V^h}}) - \sigma([\underline{\nabla_{V^h}}, (p^h)^v], \vec{W}).$$

Since \vec{W} is constant on the fiber of T^*M , one can easily show

$$(p^h)^v(\sigma(\vec{W}, \underline{\nabla_{V^h}})) - \sigma([\underline{\nabla_{V^h}}, (p^h)^v], \vec{W}) = 0.$$

Then,

$$\sigma([\vec{W}, \underline{\nabla_{V^h}}], (p^h)^v) = \sigma([\vec{W}, (p^h)^v], \underline{\nabla_{V^h}}) = g(\nabla W, v^h).$$

Hence, the required identity follows and the lemma is proved. \square

As a direct consequence of the last lemma, we have

$$\Omega^c((\text{pr} \circ \pi)_*([\vec{W}, \underline{\nabla_{V^h}}]), v^h) = \frac{1}{\|p^h\|^2} g(J^c p^h, v^h) g(v^h, \nabla W), \\ \sigma([\vec{W}, \underline{\nabla_{V^h}}], (p^h)^v) = g(v^h, \nabla W).$$

As a result of above calculations, we get

$$T_4 = -\text{Hess } W(v^h, v^h) - \frac{1}{\|p^h\|^2} g(J^c p^h, v^h) g(v^h, \nabla W) - \frac{1}{\|p^h\|^2} (g(v^h, \nabla W))^2.$$

Step 5 First of all, we show the following

Lemma 8. *The following identity holds.*

$$[\vec{W}, (J^c V^h)^v] = \frac{1}{\|p^h\|^2} g(v^h, \nabla W) (J p^h)^v.$$

Proof. As $\vec{W} = -(\nabla W)^v$ is constant on the fiber of T^*M , we can proceed with the following calculations

$$(61) \quad ([\vec{W}, (J^c V^h)^v])^h = (\text{pr} \circ \pi)_*([\vec{W}, \underline{\nabla_{J^c V^h}}]) = J^c \left((\text{pr} \circ \pi)_*([\vec{W}, \underline{\nabla_{V^h}}]) \right).$$

Substituting the identity of Lemma 7 into the last identity, we get

$$([\vec{W}, (J^c V^h)^v])^h = \frac{1}{\|p^h\|^2} g(v^h, \nabla W) J^c p^h,$$

and then the required identity follows. \square

As a direct consequence, we have

$$\sigma([\vec{W}, (J^c V^h)^v], \underline{\nabla_{v^h}}) = -\frac{1}{\|p^h\|^2} g(v^h, \nabla W) g(J^c p^h, v^h).$$

Furthermore, since $[\vec{W}, (p^h)^v] = \vec{W} = -(\nabla W)^v$, then

$$\sigma([\vec{W}, (p^h)^v], \underline{\nabla_{v^h}}) = g(\nabla W, v^h).$$

Therefore,

$$T_5 = -\frac{1}{\|p^h\|^2} (g(v^h, \nabla W))^2.$$

Combining the results of Step 1-5 and using the fact $\|p^h\|^2 = 2(c_0 + W)$, we get the expression of the reduced curvature maps, as shown in Theorem 2.

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